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Energy Decay and Exact Controllability for the Petrovsky Equation in a Bounded Domain

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For a higher dimensional Petrovsky equation in a bounded domain with linear damping and rigid homogeneous boundary conditions, the uniformly exponential energy decay is proved by a priori estimates and analysis of Lyapunov-like functional. The global exact controllability is shown by the energy decay result and time-reverse technique. For the Petrovsky equation with cubic nonlinear damping, it is proved that for any given energy bound of initial data there exists a choice of damping coefficients such that the nonlinear semigroup of solutions converges to zero strongly and uniformly.

1. INTRODUCTION

In connection with large space structures, the one-dimensional Petrovsky equation with boundary control has been extensively studied, cf. [1–5]. As for the higher dimensional Petrovsky equation, there have been some significant results on the optimal control (cf. [6]), boundary controllability and stabilization of vibrating plate with static boundary input (cf. [7, 8]), and dynamical boundary input (cf. [5, 9]). However, in comparison with the wave equation, there still remain many open problems for the higher dimensional Petrovsky equation.

Although the Petrovsky equation has some common features with the wave equation, such as that it is time-reversible so null-controllability implies the global controllability, the former differs from the latter in many aspects, e.g., there are no finite characteristic velocities for the Petrovsky equation, the well-posed boundary conditions are much involved, and the energy functions consistent with the elasticity theory (in the case of dimension $n \leq 3$) are more complicated, etc.

In this work, energy decay rate and controllability for the Petrovsky equation in a higher dimensional bounded domain with homogeneous

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boundary conditions will be considered. The obtained results can be regarded in some sense as the generalization of the corresponding results for the wave equation to the Petrovsky equation.

2. Uniform Exponential Energy Decay with Linear Damping Feedback

Let Ω be a bounded, open, and connected domain in $\mathbb{R}^n$ with piecewise C^∞ regular boundary Γ. Assume that Ω is locally located on the one side of Γ and that Γ has at most finite cusp points where the cone condition is satisfied.

Here we consider the Petrovsky equation with distributed control and rigid homogeneous boundary condition,

$$\frac{\partial^2 w}{\partial t^2} + \Delta^2 w = f(t, x), \qquad x \in \Omega, t \geq 0. \tag{1}$$

One can choose a damping feedback control to get the following equation with initial-boundary conditions,

$$\frac{\partial^2 w}{\partial t^2} + a(x)\frac{\partial w}{\partial t} + \Delta^2 w = 0, \qquad x \in \Omega, t \geq 0, \tag{2}$$

$$w(t, x) = \frac{\partial w}{\partial n}(t, x) \equiv 0, \qquad x \in \Gamma, t \geq 0, \tag{3}$$

$$\begin{aligned} w(0, x) &= w_0(x), \qquad x \in \Omega, \\ w_t(0, x) &= v_0(x), \qquad x \in \Omega, \end{aligned} \tag{4}$$

where $a(x) \geq a_0 > 0$ a.e. in Ω, and $a(\cdot) \in L^\infty(\Omega)$. Define a real product Hilbert space H by

$$H = H_0^2(\Omega) \times L^2(\Omega), \tag{5}$$

with the usual Sobolev space structure.

Lemma 1. *The "energy inner product" defined by*

$$\left\langle \begin{pmatrix} w_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ v_2 \end{pmatrix} \right\rangle = \int_\Omega (\Delta w_1 \Delta w_2 + v_1 v_2)\, dx \tag{6}$$

is equivalent to the inherent inner product of H.

Proof. From [14, Chap. 4, p. 107], the space

$$Z = \{u | u \in L^2(\Omega) \text{ and } \Delta u \in L^2(\Omega)\} \tag{7}$$

with the norm

$$\|u\|_Z = \left(\int_\Omega \left[|u|^2 + |\Delta u|^2 \right] dx \right)^{1/2} \tag{8}$$

is a Hilbert space. Now since $C_0^\infty(\Omega) \subset Z$, and (cf. [14, Chap. 4, p. 108]) the closure of $C_0^\infty(\Omega)$ in Z is just $H_0^2(\Omega)$, it follows that $(H_0^2(\Omega), \|\cdot\|_Z)$ is a closed subspace of Z, so that $(H_0^2(\Omega), \|\cdot\|_Z)$ is a Hilbert space.

Denote simply by $H_0^2(\Omega)$ the usual Sobolev space with the topology induced by the $H^2(\Omega)$-inner product. Define I: $(H_0^2(\Omega), \|\cdot\|_Z) \to H_0^2(\Omega)$ to be the identity mapping. Obviously I is a bijection. By the inverse operator theorem, I is a bounded linear operator, so that there is a constant $c_1 > 0$ such that

$$\|u\|_{H_0^2(\Omega)} \le c_1 \|u\|_Z, \qquad \forall u \in H_0^2(\Omega). \tag{9}$$

On the other hand, it is easy to see that there is a constant $c_2 > 0$ such that

$$\|u\|_Z \le c_2 \|u\|_{H_0^2(\Omega)}, \qquad \forall u \in H_0^2(\Omega). \tag{10}$$

Therefore the conclusion is valid. □

Define operator A: $\mathscr{D}(A)(\subset L^2(\Omega)) \to L^2(\Omega)$ by

$$Au = -\Delta^2 u, \qquad \mathscr{D}(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{11}$$

It can be verified that A is a selfadjoint, maximal dissipative, and coercively negative operator with compact resolvent.

Define then the operator $\mathscr{A}$: $\mathscr{D}(\mathscr{A})(\subset H) \to H$ by

$$\mathscr{A}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & -a(x)I \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix}, \qquad \text{with } \mathscr{D}(\mathscr{A}) = \mathscr{D}(A) \times H_0^2(\Omega). \tag{12}$$

Then one can prove the following properties.

LEMMA 2. *$\mathscr{A}$ is a densely defined and closed operator in H. $\mathscr{A}$ admits a compact resolvent $\mathscr{A}^{-1} \in \mathscr{L}(H)$. $\mathscr{A}$ is the infinitesimal generator of a C_0 contraction semigroup $T(t)$ $(t \ge 0)$ on H.*

Proof. $\mathscr{A} = \mathscr{A}_1 + B_1$ with

$$\mathscr{A}_1 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} : \mathscr{D}(\mathscr{A}) \to H \qquad \text{and} \qquad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & -a(x)I \end{pmatrix} \in \mathscr{L}(H). \tag{13}$$

Based on the above-mentioned properties of A and semigroup theory, we can prove that $\mathscr{A}_1$ is a densely defined and closed operator in H, that $\mathscr{A}_1$ admits a compact resolvent, and that $\mathscr{A}_1$ generates a C_0 unitary group $S(t)$ on H. Note that $\mathscr{A}$ is a bounded perturbation of $\mathscr{A}_1$, so $\mathscr{A}$ is a densely defined and closed operator. Therefore, $\mathscr{A}$ has also a compact resolvent. Since $a(x) \geq a_0 > 0$, B_1 is dissipative, so $T(t)$ is a C_0 contraction semigroup. □

Therefore the abstract Cauchy problem:

$$\frac{d}{dt}\begin{pmatrix} w(t,\cdot) \\ v(t,\cdot) \end{pmatrix} = \mathscr{A}\begin{pmatrix} w(t,\cdot) \\ v(t,\cdot) \end{pmatrix},$$
$$\begin{pmatrix} w(0,\cdot) \\ v(0,\cdot) \end{pmatrix} = \begin{pmatrix} w_0(\cdot) \\ v_0(\cdot) \end{pmatrix} \in H, \tag{14}$$

formulated from the feedback system (2)–(3)–(4) admits a unique solution

$$\begin{pmatrix} w(t,\cdot) \\ v(t,\cdot) \end{pmatrix} = T(t)\begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \in C([0,\infty); H). \tag{15}$$

THEOREM 3. *Let $w(t,x)$ be the solution of* (2) *with the boundary condition* (3) *and initial date $(w_0, v_0) \in H$. Then there exist constants $M > 0$ and $\beta > 0$, such that for all $(w_0, v_0) \in H$,*

$$\left(\int_\Omega \left[\left(\frac{\partial w}{\partial t}\right)^2 + |\Delta w|^2\right] dx\right)^{1/2} \triangleq \left\| T(t)\begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E$$
$$\leq Me^{-\beta t}\left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E, \qquad t \geq 0, \tag{16}$$

where $\|\cdot\|_E$ is the "energy norm" of the space H.

Proof. First we assume that $(w_0, v_0) \in \mathscr{D}(\mathscr{A}^\infty)$, which is dense in the space H. Since $T(t)\begin{pmatrix} w_0 \\ v_0 \end{pmatrix}$ is a strong solution of (14), we have $\partial w/\partial t \in L^2(\Omega)$.

Multiply (2) by $\partial w/\partial t$, integrate it, and use Green's formula twice to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega a(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega \Delta^2 w\frac{\partial w}{\partial t}\, dx\\
&\quad= \frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega a(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Gamma \frac{\partial}{\partial n}(\Delta w)\frac{\partial w}{\partial t}\, dx\\
&\qquad - \int_\Omega \nabla(\Delta w)\nabla\left(\frac{\partial w}{\partial t}\right) dx\\
&\quad= \frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega a(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Gamma \frac{\partial}{\partial n}(\Delta w)\frac{\partial w}{\partial t}\, dx\\
&\qquad - \int_\Gamma \Delta w\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial t}\right) dx + \int_\Omega \Delta w\cdot\Delta\left(\frac{\partial w}{\partial t}\right) dx\\
&\quad= \frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega a(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega \Delta w\cdot\Delta\left(\frac{\partial w}{\partial t}\right) dx\\
&\quad= \frac{1}{2}\frac{d}{dt}\int_\Omega\left[\left(\frac{\partial w}{\partial t}\right)^2 + |\Delta w|^2\right] dx + \int_\Omega a(x)\left(\frac{\partial w}{\partial t}\right)^2 dx = 0. \qquad (17)
\end{aligned}$$

On the other hand, multiply (2) by Θw with $\Theta > 0$ a constant to obtain

$$\Theta\left\{\frac{d}{dt}\left[\frac{\partial w}{\partial t}w\right] - \left(\frac{\partial w}{\partial t}\right)^2 + \frac{1}{2}\frac{d}{dt}\left[a(x)w^2\right] + \Delta^2 w\cdot w\right\} = 0. \qquad (18)$$

Then integrate (18) over Ω: we have

$$\Theta\left\{\frac{d}{dt}\int_\Omega \frac{\partial w}{\partial t}w\, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega a(x)w^2\, dx - \int_\Omega\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_\Omega |\Delta w|^2\, dx\right\} = 0. \qquad (19)$$

Now summing up (17) and (19), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left[\left(\frac{\partial w}{\partial t}\right)^2 + |\Delta w|^2 + 2\Theta\frac{\partial w}{\partial t}w + \Theta a(x)w^2\right] dx\\
&\qquad + \int_\Omega\left[a(x)\left(\frac{\partial w}{\partial t}\right)^2 - \Theta\left(\frac{\partial w}{\partial t}\right)^2 + \Theta|\Delta w|^2\right] dx = 0. \qquad (20)
\end{aligned}$$

Denote by

$$R(t;\Theta) = \frac{1}{2}\int_\Omega\left[\left(\frac{\partial w}{\partial t}\right)^2 + |\Delta w|^2 + 2\Theta\frac{\partial w}{\partial t}w + \Theta a(x)w^2\right]dx,$$

$$N(t;\Theta) = \int_\Omega\left[a(x)\left(\frac{\partial w}{\partial t}\right)^2 - \Theta\left(\frac{\partial w}{\partial t}\right)^2 + \Theta|\Delta w|^2\right]dx. \tag{21}$$

Then (20) can be written as

$$\frac{d}{dt}R(t;\Theta) + N(t;\Theta) = 0. \tag{22}$$

Let $a_1 > 0$ and $c > 0$ be constants such that

$$\|a(\cdot)\|_{L^\infty(\Omega)} \leq a_1 \qquad \text{and} \qquad \int_\Omega |w|^2\,dx \leq c\int_\Omega |\Delta w|^2\,dx, \qquad \text{for } w \in H_0^2(\Omega), \tag{23}$$

where the second inequality follows from Lemma 1. Thus it follows that

$$\left|\Theta\int_\Omega\left[2\frac{\partial w}{\partial t}w + a(x)w^2\right]dx\right| \leq \Theta\int_\Omega\left|\frac{\partial w}{\partial t}\right|^2 dx + \Theta(1+a_1)c\int_\Omega|\Delta w|^2\,dx$$
$$\leq \max\{\Theta, \Theta(1+a_1)c\}\int_\Omega\left[\left|\frac{\partial w}{\partial t}\right|^2 + |\Delta w|^2\right]dx. \tag{24}$$

Let

$$0 < \Theta \leq \frac{1}{2\max\{1, c(1,a_1)\}} = \Theta_1. \tag{25}$$

Then we have

$$\frac{1}{4}\int_\Omega\left[\left|\frac{\partial w}{\partial t}\right|^2 + |\Delta w|^2\right]dx \leq R(t;\Theta) \leq \int_\Omega\left[\left|\frac{\partial w}{\partial t}\right|^2 + |\Delta w|^2\right]dx. \tag{26}$$

Furthermore, if we choose

$$0 < \Theta \leq \min\left(\Theta_1, \frac{a_0}{2}\right), \tag{27}$$

then it follows that

$$N(t;\Theta) \geq \int_\Omega \left[a_0 \left| \frac{\partial w}{\partial t} \right|^2 - \Theta \left| \frac{\partial w}{\partial t} \right|^2 + \Theta |\Delta w|^2 \right] dx$$

$$\geq \min\left(\frac{a_0}{2}, \Theta \right) \int_\Omega \left[\left| \frac{\partial w}{\partial t} \right|^2 + |\Delta w|^2 \right] dx. \tag{28}$$

Let $\hat{\Theta} = \hat{c} = \min(\Theta_1, a_0/2)$, then we obtain

$$N(t;\hat{\Theta}) \geq \hat{c} \int_\Omega \left[\left| \frac{\partial w}{\partial t} \right|^2 + |\Delta w|^2 \right] dx. \tag{29}$$

From (26), (29), and $\hat{\Theta} = \min(\Theta_1, a_0/2)$, it follows that

$$N(t;\hat{\Theta}) \geq \hat{c} \int_\Omega \left[\left| \frac{\partial w}{\partial t} \right|^2 + |\Delta w|^2 \right] dx \geq \hat{c} R(t;\hat{\Theta}), \tag{30}$$

and, in view of (22), that

$$\frac{d}{dt} R(t;\hat{\Theta}) + \hat{c} R(t;\hat{\Theta}) \leq \frac{d}{dt} R(t;\hat{\Theta}) + N(t;\hat{\Theta}) = 0. \tag{31}$$

Thus

$$R(t;\hat{\Theta}) \leq R(0;\hat{\Theta}) e^{-\hat{c}t} \qquad (t \geq 0), \tag{32}$$

and by (26),

$$\left(\int_\Omega \left[\left| \frac{\partial w}{\partial t} \right|^2 + |\Delta w|^2 \right] dx \right)^{1/2} = \left\| T(t) \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E \leq 2 |R(0, \hat{\Theta})|^{1/2} e^{-\hat{c}t/2}$$

$$\leq 2 \exp\left(-\frac{\hat{c}}{2} t \right) \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E, \qquad \forall t \geq 0, \tag{33}$$

for each $\binom{w_0}{v_0} \in \mathscr{D}(\mathscr{A}^\infty)$. However, since $T(t)$ $(t \geq 0)$ is a contraction semigroup, by the denseness of $\mathscr{D}(\mathscr{A}^\infty)$ in H, it follows that (33) holds for each $\binom{w_0}{v_0} \in H$. Therefore (16) is valid. □

3. Exact Controllability

Using the exponential energy decay shown by Theorem 3 and the time-reversible property, we can easily prove the exact controllability result as follows.

Theorem 4. *For any given initial state $(w_0, v_0) \in H$, there exists a control $f(t,x) \in C([0,T]; L^2(\Omega))$ to steer the system (1)–(3)–(4) to the null state $(0,0) \in H$ at time T, provided $T > 0$ is large enough.*

Proof. Let $T > 0$ be large enough such that

$$Me^{-\beta t} < 1. \tag{34}$$

where M and β are constants appearing in (16). For $0 \le t \le T$, take a control

$$f_1(t,x) = -a(x)\frac{\partial w}{\partial t}(t,x), \qquad \text{with } a(x) \equiv a_0 > 0, \tag{35}$$

which is a distributed damping feedback. The solution of (1)–(3)–(4) with this feedback control $f_1(t,x)$ is denoted by $w_1(t,x)$ with $v_1(t,x) = (\partial w_1/\partial t)(t,x)$. Thus

$$\left\| \begin{pmatrix} w_1(T,\cdot) \\ v_1(T,\cdot) \end{pmatrix} \right\|_E \le Me^{-\beta t} \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E. \tag{36}$$

On the other hand, let

$$f_2(t,x) = a(x)\frac{\partial w}{\partial t}(t,x), \qquad \text{with } a(x) \equiv a_0 > 0. \tag{37}$$

With the time-reverse, we see that the solution of (1)–(3)–(4) with this feedback control $f_2(t,x)$, denoted by $w_2(t,x)$ with $v_2(t,x) = (\partial w_2/\partial t)(t,x)$, satisfies

$$\left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E \le Me^{-\beta T} \left\| \begin{pmatrix} w_2(T,\cdot) \\ v_2(T,\cdot) \end{pmatrix} \right\|_E. \tag{38}$$

For any given initial state $\binom{w_0}{v_0}$, let

$$\begin{pmatrix} w_T \\ v_T \end{pmatrix} = \begin{pmatrix} w_1(T,\cdot) \\ v_1(T,\cdot) \end{pmatrix} \qquad \text{with} \begin{pmatrix} w_1(0,\cdot) \\ v_1(0,\cdot) \end{pmatrix} = \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \tag{39}$$

and

$$\begin{pmatrix} w^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} w_2(0,\cdot) \\ v_2(0,\cdot) \end{pmatrix} \quad \text{with} \begin{pmatrix} w_2(T,\cdot) \\ v_2(T,\cdot) \end{pmatrix} = \begin{pmatrix} w_T \\ v_T \end{pmatrix}. \tag{40}$$

Take $f(t,x) = f_1(t,x) - f_2(t,x)$, the corresponding solution of (1)–(3) with initial condition

$$\begin{pmatrix} w(0,\cdot) \\ v(0,\cdot) \end{pmatrix} = \begin{pmatrix} w_0 - w^0 \\ v_0 - v^0 \end{pmatrix} \tag{41}$$

will satisfy automatically

$$\begin{pmatrix} w(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{42}$$

The only thing to be proved is

$$\left\{ \begin{pmatrix} w_0 - w^0 \\ v_0 - v^0 \end{pmatrix} \in H \,\middle|\, \forall \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \in H \right\} = H. \tag{43}$$

By the uniqueness, we have a linear mapping $P\colon \binom{w_0}{v_0} \to \binom{w^0}{v^0}$ with

$$\|P\|_{\mathscr{L}(H_E)} \leq (Me^{-\beta t})^2 < 1. \tag{44}$$

where H_E represents the space H with the energy norm. Hence it follows that $I - P \in \mathscr{L}(H_E)$ is invertible so that (43) holds. □

Remark 5. From the argument in Theorem 4, the time $T > 0$ for the exact null controllability can be uniform for all the initial states.

Remark 6. By the time-reverse process, this exact null controllability implies the global exact controllability with uniform $T > 0$.

4. Petrovsky Equation with Cubic Nonlinearity

Let $f(t,x)$ be a nonlinear damping feedback control, given by

$$f(t,x) = -a_1 \frac{\partial w}{\partial t} - a_2 \left(\frac{\partial w}{\partial t} \right)^3, \quad \text{with } a_1, a_2 > 0 \text{ constants}. \tag{45}$$

Then we shall investigate the following nonlinear Petrovsky equation,

$$\frac{\partial^2 w}{\partial t^2} + a_1 \frac{\partial w}{\partial t} + a_2 \left(\frac{\partial w}{\partial t} \right)^3 + \Delta^2 w = 0, \tag{46}$$

with the homogeneous boundary condition (3) and initial condition (4).

Define a nonlinear differential operator G: $\mathscr{D}(G) \to H$ by

$$\mathscr{D}(G) = \mathscr{D}(A) \times \left(H_0^2(\Omega) \cap L^6(\Omega)\right),$$

$$G\binom{w}{v} = \begin{pmatrix} v \\ -\Delta^2 w - a_1 v - a_2 v^3 \end{pmatrix}. \tag{47}$$

Equation (46) can be written in the form

$$\frac{d}{dt}\binom{w}{v} = G\binom{w}{v}. \tag{48}$$

From the Sobolev embedding theorem,

$$H_0^m(\Omega) \subset H^m(\Omega) \subset L^p(\Omega) \qquad \text{for } \frac{1}{p} \geq \frac{1}{2} - \frac{m}{n}, \tag{49}$$

it follows that

$$H_0^2(\Omega) = H_0^2(\Omega) \cap L^6(\Omega) \qquad \text{for } n \leq 6. \tag{50}$$

For simplicity and in view of practical purposes, we assume in this section that $n \leq 6$. So $\mathscr{D}(G) = \mathscr{D}(\mathscr{A})$, cf. (12).

First we prove the following result.

THEOREM 7. *G: $\mathscr{D}(G)$ $(= \mathscr{D}(\mathscr{A})$ for $n \leq 6) \to H$ is a maximal monotone operator in H, and it generates a strongly continuous nonlinear contraction semigroup on H.*

Proof. For any $\binom{w}{v} \in \mathscr{D}(G) = \mathscr{D}(\mathscr{A})$, we have, by the skew-adjointness of $\mathscr{A}_1$,

$$\left\langle G\binom{w}{v}, \binom{w}{v} \right\rangle = -\int_\Omega \left(a_1 v^2 + a_2 v^4\right) dx \leq 0. \tag{51}$$

Thus G is monotone (or called dissipative) operator.

By Minty's theorem (cf. [15]), G is maximal monotone provided

$$\operatorname{Ran}(I - G) = H. \tag{52}$$

For any given $\binom{p}{q} \in H$, we show that there exists an element $\binom{w}{v} \in \mathscr{D}(G)$

such that

$$(I-G)\binom{w}{v}=\binom{p}{q}. \tag{53}$$

Equation (53) amounts to equations

$$\begin{gathered} w-v=p \\ \Delta^2 w+(1+a_1)v+a_2v^3=q. \end{gathered} \tag{54}$$

Substituting the first relation into the second one, we obtain an equation with respect to v, i.e.,

$$\Delta^2 v+(1+a_1)v+a_2v^3=q-\Delta^2 p, \tag{55}$$

where $p\in H_0^2(\Omega)$ implies that $\Delta^2 p\in H^{-2}(\Omega)$.

Consider now the nonlinear elliptic operator

$$Q(v)=\Delta^2 v+(1+a_1)v+a_2v^3\colon H_0^2(\Omega)\to H^{-2}(\Omega). \tag{56}$$

Here $H_0^2(\Omega)$ is a reflexive Banach space, and it is easy to verify that Q: $H_0^2(\Omega)\to H^{-2}(\Omega)$ is monotone, everywhere defined, demicontinuous, and coercive, i.e.,

$$\lim_{\|v\|_{H^2}\to\infty}\frac{(Q(v),v)_{\text{dual}}}{\|v\|_{H^2}}=+\infty. \tag{57}$$

By Theorem 4.3 of [16, p. 59], it follows that

$$\operatorname{Ran} Q=H^{-2}(\Omega). \tag{58}$$

Furthermore, the coercivity of $Q(v)$ implies that Q is also an injection. Therefore Q^{-1}: $H^{-2}(\Omega)\to H_0^2(\Omega)$ exists and (55) is solvable for any given $\binom{p}{q}\in H$.

Then let $w=v+p$. It follows from (55) that

$$w\in H_0^2(\Omega) \qquad \text{and} \qquad \Delta^2 w=q-(1+a_1)v-a_2v^3\in L^2(\Omega). \tag{59}$$

By the null extension, which is feasible due to $w\in H_0^2(\Omega)$, and the fact that

$$u\in H^{2m}(\mathbb{R}^n) \qquad \text{if and only if } (1-\Delta)^m u\in L^2(\mathbb{R}^n), \tag{60}$$

we have then

$$w\in H^4(\Omega)\cap H_0^2(\Omega). \tag{61}$$

Therefore, (54) is solvable in $\mathscr{D}(G)$ for any given $\binom{p}{q} \in H$. Thus (52) is valid and G is actually a maximal monotone operator in H.

Apply the generation theorem for nonlinear contraction semigroups (cf. [15]), this operator G generates a strongly continuous nonlinear semigroup $S(t)$ $(t \geq 0)$ of contractions on H. □

COROLLARY 8. *The solution of the nonlinear Petrovsky equation* (46) *with* (3) *and* (4) *exists and is unique for* $t \geq 0$, *and such that*

$$w \in L^\infty\big(0, \infty; H_0^2(\Omega)\big) \qquad w_t \in L^\infty\big(0, \infty; L^2(\Omega)\big). \tag{62}$$

Next we show the energy decay estimates for the solutions of Eq. (46).

THEOREM 9. *Let* $f(t,x)$ *be a nonlinear feedback* (45) *with* a_1, a_2 *to be chosen. For any given* $\rho > 0$, *there exist a pair* $\{a_1 > 0, a_2 > 0\}$ *and constants* $K, \gamma > 0$, *such that*

$$\left\| \begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot) \end{pmatrix} \right\|_E^2 \leq K e^{-\gamma t} \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E^2, \qquad \forall t \geq 0, \tag{63}$$

for all $\binom{w_0}{v_0} \in \mathscr{D}(G)$ *with* $\|\binom{w_0}{v_0}\|_E \leq \rho$.

Proof. Multiply (46) by $\partial w/\partial t + \Theta w$ with $\Theta > 0$ a constant and then integrate over Ω; we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \left[\left(\frac{\partial w}{\partial t}\right)^2 + |\Delta w|^2 + 2\Theta \frac{\partial w}{\partial t} w + \Theta a_1 w^2\right] dx \\ &\quad + \int_\Omega \left[a_1 \left(\frac{\partial w}{\partial t}\right)^2 + a_2 \left(\frac{\partial w}{\partial t}\right)^4 \right. \\ &\qquad\qquad \left. + \Theta |\Delta w|^2 - \Theta \left(\frac{\partial w}{\partial t}\right)^2 + \Theta a_2 \left(\frac{\partial w}{\partial t}\right)^3 w \right] dx = 0. \end{aligned} \tag{64}$$

Denote by

$$\begin{aligned} \tilde{R}(t,\Theta) &= \frac{1}{2}\int_\Omega \left[\left(\frac{\partial w}{\partial t}\right)^2 + |\Delta w|^2 + 2\Theta \frac{\partial w}{\partial t} w + \Theta a_1 w^2\right] dx, \\ \tilde{N}(t,\Theta) &= \int_\Omega \left[a_1 \left(\frac{\partial w}{\partial t}\right)^2 + a_2 \left(\frac{\partial w}{\partial t}\right)^4 + \Theta |\Delta w|^2 \right. \\ &\qquad\qquad \left. - \Theta \left(\frac{\partial w}{\partial t}\right)^2 + \Theta a_2 \left(\frac{\partial w}{\partial t}\right)^3 w \right] dx. \end{aligned} \tag{65}$$

Then (64) can be written as

$$\frac{d}{dt}\tilde{R}(t,\Theta) + \tilde{N}(t,\Theta) = 0, \qquad t \geq 0. \tag{66}$$

Similar to (24) through (26), we have then

$$\frac{1}{4}\int_\Omega\left[\left|\frac{\partial w}{\partial t}\right|^2 + |\Delta w|^2\right]dx \leq \tilde{R}(t,\Theta) \leq \int_\Omega\left[\left|\frac{\partial w}{\partial t}\right|^2 + |\Delta w|^2\right]dx, \tag{67}$$

provided $0 < \Theta \leq \Theta_1$, where Θ_1 is given by (25).

On the other hand, using Hölder's inequality we get

$$\left|\Theta a_2\int_\Omega\left(\frac{\partial w}{\partial t}\right)^3 w\,dx\right| \leq \Theta a_2\left\{\frac{3}{4}\int_\Omega\left(\frac{\partial w}{\partial t}\right)^4 dx + \frac{1}{4}\int_\Omega w^4\,dx\right\}. \tag{68}$$

Let

$$\Theta_2 = \min\left\{\Theta_1, \frac{a_1}{2}\right\} = \min\left\{\frac{1}{2}, \frac{1}{2c(1+a_1)}, \frac{a_1}{2}\right\}. \tag{69}$$

Then we have

$$\begin{aligned}\tilde{N}(t,\Theta_2) &\geq \int_\Omega\left[(a_1 - \Theta_2)\left(\frac{\partial w}{\partial t}\right)^2 + a_2\left(1 - \frac{3\Theta_2}{4}\right)\left(\frac{\partial w}{\partial t}\right)^4\right. \\ &\qquad\qquad \left. + \Theta_2|\Delta w|^2 - \frac{\Theta_2 a_2}{4}w^4\right]dx \\ &\geq \int_\Omega\left[\frac{a_1}{2}\left(\frac{\partial w}{\partial t}\right)^2 + \frac{a_2}{2}\left(\frac{\partial w}{\partial t}\right)^4 + \Theta_2|\Delta w|^2 - \frac{a_2}{8}w^4\right]dx \\ &\geq \Theta_2\int_\Omega\left[\left|\frac{\partial w}{\partial t}\right|^2 + |\Delta w|^2\right]dx - \frac{a_2}{8}\int_\Omega w^4\,dx. \end{aligned}\tag{70}$$

From (66), (67), and (70), it follows that

$$\begin{aligned}&\frac{d}{dt}\tilde{R}(t,\Theta_2) + \Theta_2\tilde{R}(t,\Theta_2) - \frac{a_2}{8}\int_\Omega w^4\,dx \\ &\qquad \leq \frac{d}{dt}\tilde{R}(t,\Theta_2) + \tilde{N}(t,\Theta_2) = 0, \qquad \forall t \geq 0. \end{aligned}\tag{71}$$

Then it follows that, by solving the differential inequality (71),

$$\frac{1}{4}\left\|\begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot)\end{pmatrix}\right\|_E^2 \le \tilde{R}(T,\Theta_2) \le e^{-\Theta_2 t}\tilde{R}(0,\Theta_2) + \frac{a_2}{8}\int_0^t\int_\Omega e^{\Theta_2(\tau-t)}w^4\,dx\,d\tau$$
$$\le e^{-\Theta_2 t}\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 + \frac{a_2}{8}\int_0^t\int_\Omega e^{\Theta_2(\tau-t)}w^4\,dx\,d\tau, \qquad \forall t \ge 0. \tag{72}$$

Thus we obtain

$$\left\|\begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot)\end{pmatrix}\right\|_E^2 \le 4e^{-\Theta_2 t}\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 + \frac{a_2}{2}\int_0^t\int_\Omega e^{\Theta_2(\tau-t)}w^4\,dx\,d\tau, \qquad \forall t \ge 0. \tag{73}$$

By the assumption $n \le 6$ in this section and (49) and (50), we have $H_0^2(\Omega) \subset L^4(\Omega)$, so that there exists a constant $\delta > 0$, such that

$$\int_\Omega w^4 dx \le \delta\left(\int_\Omega |\Delta w|^2\,dx\right) \le \delta\left\|\begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot)\end{pmatrix}\right\|_E^2. \tag{74}$$

Since G generates a contraction semigroup on H, we have

$$\left\|\begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot)\end{pmatrix}\right\|_E^2 \le C_1\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 \qquad \text{with } C_1 > 0 \text{ a constant.} \tag{75}$$

Here C_1 may not be 1 because E-norm is an equivalent norm for H.

Substituting (74) and (75) into (73), we get

$$g(t) \le 4\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 + \frac{a_2\delta C_1}{2}\int_0^t\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 g(\tau)\,d\tau, \qquad t \ge 0, \tag{76}$$

where

$$g(t) = e^{\Theta_2 t}\left\|\begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot)\end{pmatrix}\right\|_E^2. \tag{77}$$

By the Gronwall inequality, it follows that

$$g(t) \le 4\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 \exp\left(\frac{a_2\delta C_1}{2}\left\|\begin{pmatrix} w_0 \\ v_0\end{pmatrix}\right\|_E^2 t\right), \qquad t \ge 0, \tag{78}$$

so that

$$\left\| \begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot) \end{pmatrix} \right\|_E^2 \le 4 \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E^2 \exp\left\{ \left(\frac{a_2 \delta C_1}{2} \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E^2 - \Theta_2 \right) t \right\}, \qquad t \ge 0. \tag{79}$$

For any given $\rho > 0$, we fix a constant $a_1 > 0$ arbitrarily, then there exists a constant $a_2 > 0$, such that

$$a_2 < \frac{2\Theta_2}{\delta C_1 \rho^2} = \frac{1}{\delta C_1 \rho^2} \min\left\{ 1, \frac{1}{c(1+a_1)}, a_1 \right\}. \tag{80}$$

With this $a_2 > 0$, the nonlinear feedback (45) makes the solution decay,

$$\left\| \begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot) \end{pmatrix} \right\|_E^2 \le 4e^{-\gamma t} \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_E^2, \qquad \forall t \ge 0, \tag{81}$$

for all $\binom{w_0}{v_0} \in \mathscr{D}(G)$ with $\|\binom{w_0}{v_0}\|_E \le \rho$, where

$$\gamma = \Theta_2 - \frac{1}{2} a_2 \delta C_1 \rho^2 > 0. \tag{82}$$

Thus the conclusion is valid. □

THEOREM 10. *For any given $\rho > 0$, there exist constants $a_1 > 0, a_2 > 0$ such that the nonlinear feedback control $f(t,x)$ with this pair of coefficients $\{a_1, a_2\}$ will make the solution of* (46) *strongly converge to zero in H, i.e.*,

$$\lim_{t \to +\infty} \left\| \begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot) \end{pmatrix} \right\|_H = 0, \tag{83}$$

for all the initial states satisfying

$$\left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_H \le \rho. \tag{84}$$

Proof. Note that in the space H, two norms $\|\cdot\|_H$ and $\|\cdot\|_E$ are equivalent. By Theorem 9, for any given $\rho > 0$, there exist $a_1 > 0$, $a_2 > 0, K > 0, \gamma > 0$, such that

$$\left\| \begin{pmatrix} w(t,\cdot) \\ w_t(t,\cdot) \end{pmatrix} \right\|_H^2 \le K e^{-\gamma t} \left\| \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_H^2, \qquad \forall t \ge 0, \tag{85}$$

for $\binom{w_0}{v_0} \in \mathscr{D}(G)$ and $\|\binom{w_0}{v_0}\|_H \le \rho + 1$.

Since $S(t)$ is a nonlinear contraction semigroup, now for any initial state $\binom{w_0}{v_0} \in H$ with $\|\binom{w_0}{v_0}\|_H \le \rho$, there exists a sequence of elements $\{\binom{w_n}{v_n}\}_{n=1}^{\infty} \subset \mathscr{D}(G)$, with $\|\binom{w_n}{v_n}\|_H \le \rho + 1$ such that $\|\binom{w_n}{v_n} - \binom{w_0}{v_0}\|_H \to 0$ as $n \to \infty$. By the fact that for each $n \ge 1$,

$$\left\| S(t)\begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_H \le \left\| S(t)\begin{pmatrix} w_n \\ v_n \end{pmatrix} \right\|_H + \left\| \begin{pmatrix} w_n \\ v_n \end{pmatrix} - \begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_H, \tag{86}$$

and (85), it holds that

$$\lim_{t \to +\infty} \left\| S(t)\begin{pmatrix} w_0 \\ v_0 \end{pmatrix} \right\|_H = 0, \tag{87}$$

which implies that (83) is valid. Moreover, it can be seen that the convergence (87) or (83) is uniform for all the initial states in any given bounded set satisfying $\|\binom{w_0}{v_o}\|_H \le \rho$ with ρ fixed. □

References

1. A. V. Balakrishnan, A mathematical formulation of a large space structure control problem, *in* "Proceedings, 24th Conference on Decision and Control, Ft. Lauderdale, FL, 1985," pp. 1989–1993.
2. W. Littman and L. Markus, Exact boundary controllability of a hybrid system of elasticity, *Arch. Rational Mech. Anal.*, **103** (3), (1988), 193–236.
3. W. Littman and L. Markus, Stabilization of a hybrid system of elasticity by feedback boundary damping, *Ann. Mat. Pura Appl.*, **152**, (1988), 281–330.
4. D. L. Russell, Mathematical models for the elastic beam and their control-theoretic implications, in "Semigroups, Theory and Applications," Vol 2, pp. 117–216, Pitman, London, 1986.
5. Y. You, "Stabilization, Optimal Control, and Inertial Manifolds of Infinite Dimensional Systems," Ph.D. thesis. University of Minnesota, 1988.
6. J. L. Lions, "Contrôle des Systèmes Distribués Singuliers," Gauthier–Villars, Paris, 1983.
7. J. E. Lagnese, Uniform boundary stabilization of plates, *in* "Proceedings, 26th CDC, Los Angeles, 1987," FA-6.
8. Z. Enrike, Controlabilite Exacte d'un Modele de Plaques Vibrantes en un Temps Arbitrairement Petit, *C.R. Acad. Sci., Paris* **304**, Ser. 1, No. 7, (1987), 173–176.
9. Y. You, Dynamical boundary control of two-dimensional Petrovsky system: Vibrating rectangular plate, *in* "Analysis and Optimization of Systems," pp. 519–530, Springer–Verlag, 1988.
10. C. S. Morawetz, Exponential decay of solutions of the wave equation, *Comm. Pure Appl. Math.* **19** (1966), 439–444.
11. C. S. Morawetz, Decay for solutions of the exterior problem for the wave equation, *Comm. Pure Appl. Math.* **28** (1975), 229–264.

12. J. LAGNESE, Decay of solutions of wave equations in a bounded region with boundary dissipation, *J. Differential Equations* **50**, No. 2 (1983), 163–182.
13. W. A. STRAUSS, Dispersal of waves vanishing on the boundary of an exterior domain, *Comm. Pure Appl. Math.* **28** (1975), 265–278.
14. J. L. LIONS, "Problèmes aux Limites dans les Equations aux Dérivées Partielles," Presses Univ. Montreal, Montreal, 1965.
15. A. HARAUX, "Nonlinear Evolution Equations—Global Behavior of Solutions," Lecture Notes in Mathematics, Vol. 841, Springer-Verlag, New York/Berlin, 1981.
16. V. BARBU AND TH. PRECUPANU, "Convexity and Optimization in Banach Spaces," Sijthoff & Noordhoff, Bucharest, Roumania, 1978.